Asymptotic behaviour of the inductance coefficient for thin conductors

Youcef Amirat, Rachid Touzani
Laboratoire de Mathématiques Appliquées, UMR CNRS 6620
Université Blaise Pascal (Clermont–Ferrand)
63177 Aubière cedex, France

Abstract

We study the asymptotic behaviour of the inductance coefficient for a thin toroidal inductor whose thickness depends on a small parameter $\varepsilon > 0$. We give an explicit form of the singular part of the corresponding potential u^ε which allows to construct the limit potential u (as $\varepsilon \to 0$) and an approximation of the inductance coefficient L^ε. We establish some estimates of the deviation $u^\varepsilon - u$ and of the error of approximation of the inductance. We show that L^ε behaves asymptotically as $\ln\varepsilon$, when $\varepsilon \to 0$.

Résumé

On étudie le comportement asymptotique du coefficient d'inductance pour un inducteur toroïdal filiforme dont l'épaisseur dépend d'un petit paramètre $\varepsilon > 0$. On donne une forme explicite de la partie singulière du potentiel associé u^ε puis on construit le potentiel limite u (quand $\varepsilon \to 0$) et on donne une approximation du coefficient d'inductance L^ε. On établit des estimations de l'écart $u^\varepsilon - u$ et de l'erreur d'approximation de l'inductance. On montre que L^ε se comporte asymptotiquement comme $\ln\varepsilon$ au voisinage de $\varepsilon = 0$.

Key Words : Asymptotic behaviour, self inductance, eddy currents, thin domain
AMS Subject Classification : 35B40, 35Q60

1 Introduction

Electrotechnical devices often involve thick conductors in which a magnetic field can be induced, and thin wires or coils, as inductors, connected to a power source generator. The problem is then to derive mathematical models which take into account the simultaneous presence of thick conductors and thin inductors. For a two–dimensional configuration where the magnetic field has only one nonvanishing component, it was shown that the eddy current equation has the Kirchhoff circuit equation as a limit problem, as the thickness of the inductor tends to zero, see [8]. For the three–dimensional case, eddy current models require the use of a relevant quantity that is the self inductance of the inductor, see [1], [2]. This number has to be evaluated *a priori* as a part of problem data. It is the purpose of the present paper to study the asymptotic behaviour of this number when the thickness of the inductor goes to zero.

Let us consider a toroidal domain of $\mathbb{R}^3$, denoted by Ω_ε, whose thickness depends on a small parameter $\varepsilon > 0$. The geometry of Ω_ε will be described in the next section. We denote by Γ_ε the boundary of Ω_ε, by n_ε the outward unit normal to Γ_ε, and by Ω'_ε the complementary of its closure, that is $\Omega'_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}_\varepsilon$. We denote by Σ a cut in the domain Ω'_ε, that is, Σ is a smooth orientable surface such that, for any $\varepsilon > 0$, $\Omega'_\varepsilon \setminus \Sigma$ is simply connected.

Let now $\boldsymbol{h}^\varepsilon$ denote the time–harmonic and complex valued magnetic field. Neglecting the displacement currents, it follows from Maxwell's equations that

$$\mathbf{curl}\, \boldsymbol{h}^\varepsilon = 0, \ \operatorname{div} \boldsymbol{h}^\varepsilon = 0 \quad \text{in } \Omega'_\varepsilon.$$

Then, by a result in [4], p. 265, $\boldsymbol{h}^\varepsilon$ may be written in the form

$$\boldsymbol{h}^\varepsilon_{|\Omega'_\varepsilon} = \nabla\varphi^\varepsilon + I^\varepsilon \nabla u^\varepsilon, \tag{1.1}$$

where I^ε is a complex number, $\varphi^\varepsilon \in W^1(\Omega'_\varepsilon)$ and satisfies

$$\Delta\varphi^\varepsilon = 0 \qquad \text{in } \Omega'_\varepsilon,$$

and u^ε is solution of :

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, & \\ \left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma = 0. & \end{cases} \tag{1.2}$$

Here $W^1(\Omega'_\varepsilon)$ is the Sobolev space

$$W^1(\Omega'_\varepsilon) = \left\{v;\ \rho v \in L^2(\Omega'_\varepsilon),\ \nabla v \in \boldsymbol{L}^2(\Omega'_\varepsilon)\right\},$$

equipped with the norm

$$\|v\|_{W^1(\Omega'_\varepsilon)} = \left(\|\rho v\|^2_{L^2(\Omega'_\varepsilon)} + \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega'_\varepsilon)}\right)^{\frac{1}{2}}, \tag{1.3}$$

where $\boldsymbol{L}^p(\Omega'_\varepsilon)$ denotes the space $L^p(\Omega'_\varepsilon)^3$ and ρ is the weight function $\rho(\boldsymbol{x}) = (1+|\boldsymbol{x}|^2)^{-\frac{1}{2}}$. Let us note here, see [4], pp. 649–651, that

$$|v|_{W^1(\Omega'_\varepsilon)} = \left(\int_{\Omega'_\varepsilon} |\nabla v|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}}$$

is a norm on $W^1(\Omega'_\varepsilon)$, equivalent to (1.3). In (1.2), n is the unit normal on Σ, and $[u^\varepsilon]_\Sigma$ (resp. $\left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma$) denotes the jump of u^ε (resp. $\dfrac{\partial u^\varepsilon}{\partial n}$) across Σ.
In (1.1), the number I^ε can be interpreted as the total current flowing in the inductor, see [2].
The inductance coefficient is then defined by the expression

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\boldsymbol{x}. \tag{1.4}$$

Our goal is to study the asymptotic behaviour of u^ε and L^ε as ε goes to zero. We first give an explicit form of the singular part of the potential u^ε which allows to construct the limit potential u (as $\varepsilon \to 0$) and an approximation of the inductance L^ε. We then prove that the deviation $\|u^\varepsilon - u\|_{W^1(\Omega'_\varepsilon)}$ and the error of approximation of L^ε is at order $O(\varepsilon^{\frac{5}{6}})$. Finally we show that the inductance coefficient L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \to 0$, and we thus recover the result stated (without proof) in [6], p. 137.
The remaining of this paper is organized as follows. In Section 2 we precise the geometry of the inductor by considering that this one is obtained by generating a toroidal domain around a closed curve, the internal radius of the torus being proportional to a small positive number ε. Section 3 states the main result and Section 4 is devoted to the proof.

2 Geometry of the domain

We consider a toroidal domain, with a small cross section. This domain may be defined as a tubular neighborhood of a closed curve. Let γ denote a closed Jordan arc of class $\mathcal{C}^3$ in $\mathbb{R}^3$, with a parametric representation defined by a function $\boldsymbol{g} : [0,1] \to \mathbb{R}^3$ satisfying

$$\boldsymbol{g}(0) = \boldsymbol{g}(1), \;\; \boldsymbol{g}'(0) = \boldsymbol{g}'(1), \;\; |\boldsymbol{g}'(s)| \geq C_0 > 0. \tag{2.1}$$

For each $s \in (0,1]$ we denote by $(\boldsymbol{t}(s), \boldsymbol{\nu}(s), \boldsymbol{b}(s))$ the Serret–Frénet coordinates at the point $\boldsymbol{g}(s)$, *i.e.*, $\boldsymbol{t}(s), \boldsymbol{\nu}(s), \boldsymbol{b}(s)$ are respectively the unit tangent vector to γ, the principal normal and the binormal, given by

$$\boldsymbol{t} = \frac{\boldsymbol{g}'}{|\boldsymbol{g}'|}, \;\; \boldsymbol{\nu} = \frac{\boldsymbol{t}'}{|\boldsymbol{t}'|}, \;\; \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{\nu}.$$

We have the following well-known Serret–Frénet formulae :

$$\boldsymbol{t}' = \kappa \boldsymbol{\nu}, \;\; \boldsymbol{\nu}' = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \;\; \boldsymbol{b}' = -\tau \boldsymbol{\nu},$$

where κ and τ denote respectively the curvature and the torsion of the arc γ. Let $\widehat{\Omega} = (0,1)^2 \times (0, 2\pi)$ and let δ denote a positive number to be chosen in a convenient way. We define, for any ε, $0 \le \varepsilon < \delta$, the mapping $\boldsymbol{F}_\varepsilon : \widehat{\Omega} \to \mathbb{R}^3$ by

$$\boldsymbol{F}_\varepsilon(s,\xi,\theta) = \boldsymbol{g}(s) + r_\varepsilon(\xi)(\cos\theta\,\boldsymbol{\nu}(s) + \sin\theta\,\boldsymbol{b}(s)),$$

where $r_\varepsilon(\xi) = (\delta-\varepsilon)\xi + \varepsilon$. We have

$$\begin{aligned}
\frac{\partial \boldsymbol{F}_\varepsilon}{\partial s} &= \boldsymbol{g}' + r_\varepsilon(\cos\theta\,\boldsymbol{\nu}' + \sin\theta\,\boldsymbol{b}') \\
&= (|\boldsymbol{g}'| - r_\varepsilon\kappa\cos\theta)\boldsymbol{t} + r_\varepsilon\tau(\cos\theta\,\boldsymbol{b} - \sin\theta\,\boldsymbol{\nu}), \\
\frac{\partial \boldsymbol{F}_\varepsilon}{\partial \xi} &= (\delta-\varepsilon)(\cos\theta\,\boldsymbol{\nu} + \sin\theta\,\boldsymbol{b}), \\
\frac{\partial \boldsymbol{F}_\varepsilon}{\partial \theta} &= r_\varepsilon(-\sin\theta\,\boldsymbol{\nu} + \cos\theta\,\boldsymbol{b}).
\end{aligned}$$

The jacobian of $\boldsymbol{F}_\varepsilon$ is therefore given by

$$J_\varepsilon(s,\xi,\theta) = (\delta-\varepsilon)a_\varepsilon(s,\xi,\theta)r_\varepsilon(\xi),$$

where

$$a_\varepsilon(s,\xi,\theta) = |\boldsymbol{g}'(s)| - r_\varepsilon(\xi)\kappa(s)\cos\theta.$$

According to (2.1), if δ is chosen such that

$$\delta|\kappa(s)| < |\boldsymbol{g}'(s)|, \qquad 0 \le s \le 1,$$

then

$$0 < C_1 \le a_\varepsilon \le C_2, \tag{2.2}$$

and the mapping $\boldsymbol{F}_\varepsilon$ is a $\mathcal{C}^1$–diffeomorphism from $\widehat{\Omega}$ into $\Lambda_\varepsilon^\delta = \boldsymbol{F}_\varepsilon(\widehat{\Omega})$.
Here and in the sequel, the quantities $C, C_1, C_2, \ldots$ denote generic positive numbers that do not depend on ε.

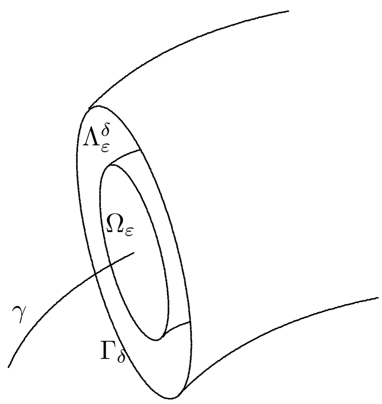

Figure 1 – A sketch of the inductor geometry

We now set, for any $0 < \varepsilon < \delta$,

$$\Omega_\delta = \Lambda_0^\delta = \boldsymbol{F}_0(\widehat{\Omega}),\ \Omega'_\delta = \mathbb{R}^3 \setminus \overline{\Omega}_\delta,\ \Omega'_\varepsilon = \mathrm{Int}(\overline{\Omega}'_\delta \cup \overline{\Lambda}^\delta_\varepsilon),\ \Omega_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}'_\varepsilon.$$

For technical reasons, we choose in the sequel $0 < \varepsilon \leq \frac{\delta}{2}$.
Given a function v on $\Lambda^\delta_\varepsilon$, we define the function $\widehat{v}$ on $\widehat{\Omega}$ by $\widehat{v} = v \circ \boldsymbol{F}_\varepsilon$. If $v \in L^p(\Lambda^\delta_\varepsilon)$, $1 \leq p \leq \infty$, then $\widehat{v} \in L^p(\widehat{\Omega})$ and we have

$$\int_{\Lambda^\delta_\varepsilon} v\, d\boldsymbol{x} = \int_{\widehat{\Omega}} \widehat{v}\,(\delta - \varepsilon)\, a_\varepsilon r_\varepsilon\, d\widehat{\boldsymbol{x}}.$$

If $v \in W^{1,p}(\Lambda^\delta_\varepsilon)$, $1 \leq p \leq \infty$, then $\widehat{v} \in W^{1,p}(\widehat{\Omega})$ and we have

$$\frac{\partial \widehat{v}}{\partial s} = \widehat{\nabla v} \cdot \frac{\partial \boldsymbol{F}_\varepsilon}{\partial s} = \widehat{\nabla v} \cdot (a_\varepsilon \boldsymbol{t} + r_\varepsilon \tau \cos\theta\, \boldsymbol{b} - \boldsymbol{r}_\varepsilon \tau \sin\theta\, \boldsymbol{\nu}), \tag{2.3}$$

$$\frac{\partial \widehat{v}}{\partial \xi} = \widehat{\nabla v} \cdot \frac{\partial \boldsymbol{F}_\varepsilon}{\partial \xi} = (\delta - \varepsilon)\widehat{\nabla v} \cdot (\cos\theta\, \boldsymbol{\nu} + \sin\theta\, \boldsymbol{b}), \tag{2.4}$$

$$\frac{\partial \widehat{v}}{\partial \theta} = \widehat{\nabla v} \cdot \frac{\partial \boldsymbol{F}_\varepsilon}{\partial \theta} = r_\varepsilon \widehat{\nabla v} \cdot (-\sin\theta\, \boldsymbol{\nu} + \cos\theta\, \boldsymbol{b}). \tag{2.5}$$

From (2.4) and (2.5) we deduce

$$\widehat{\nabla v} \cdot \boldsymbol{b} = \frac{\sin\theta}{\delta - \varepsilon} \frac{\partial \widehat{v}}{\partial \xi} + \frac{\cos\theta}{r_\varepsilon} \frac{\partial \widehat{v}}{\partial \theta}, \tag{2.6}$$

$$\widehat{\nabla v} \cdot \boldsymbol{\nu} = \frac{\cos\theta}{\delta - \varepsilon} \frac{\partial \widehat{v}}{\partial \xi} - \frac{\sin\theta}{r_\varepsilon} \frac{\partial \widehat{v}}{\partial \theta}, \tag{2.7}$$

and then, with (2.3) we get

$$\widehat{\nabla v} \cdot \boldsymbol{t} = \frac{1}{a_\varepsilon} \left(\frac{\partial \widehat{v}}{\partial s} - \tau \frac{\partial \widehat{v}}{\partial \theta} \right). \tag{2.8}$$

Therefore, for u and v in $H^1(\Lambda^\delta_\varepsilon)$,

$$\begin{aligned} \int_{\Lambda^\delta_\varepsilon} \nabla u . \nabla v\, d\boldsymbol{x} = (\delta - \varepsilon) \int_{\widehat{\Omega}} \Bigg(& \frac{r_\varepsilon}{a_\varepsilon} \frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial s} + \frac{r_\varepsilon a_\varepsilon}{(\delta - \varepsilon)^2} \frac{\partial \widehat{u}}{\partial \xi} \frac{\partial \widehat{v}}{\partial \xi} \\ & + \left(\frac{a_\varepsilon}{r_\varepsilon} + \frac{\tau^2 r_\varepsilon}{a_\varepsilon} \right) \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial \theta} \\ & - \frac{r_\varepsilon \tau}{a_\varepsilon} \left(\frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial \theta} + \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial s} \right) \Bigg) d\widehat{\boldsymbol{x}}. \end{aligned} \tag{2.9}$$

We also define the set $\widehat{\Gamma} = (0, 1) \times (0, 2\pi)$ and the mapping $\boldsymbol{G}_\varepsilon : \widehat{\Gamma} \to \mathbb{R}^3$ by

$$\boldsymbol{G}_\varepsilon(s, \theta) = \boldsymbol{g}(s) + \varepsilon(\cos\theta\, \boldsymbol{\nu}(s) + \sin\theta\, \boldsymbol{b}(s)).$$

The boundary of Ω'_ε is then represented by $\Gamma_\varepsilon = \overline{\boldsymbol{G}_\varepsilon(\widehat{\Gamma})}$. We have

$$\frac{\partial \boldsymbol{G}_\varepsilon}{\partial s} = (|\boldsymbol{g}'| - \varepsilon\kappa\cos\theta)\boldsymbol{t} + \varepsilon\tau(\cos\theta\, \boldsymbol{b} - \sin\theta\, \boldsymbol{\nu}),$$

$$\frac{\partial \boldsymbol{G}_\varepsilon}{\partial \theta} = \varepsilon(-\sin\theta\, \boldsymbol{\nu} + \cos\theta\, \boldsymbol{b}).$$

If $w \in L^2(\Gamma_\varepsilon)$, we define $\widehat{w} \in L^2(\widehat{\Gamma})$ by $\widehat{w} = w \circ \boldsymbol{G}_\varepsilon$, and we have

$$\int_{\Gamma_\varepsilon} w \, d\sigma = \int_{\widehat{\Gamma}} \widehat{w} \, \varepsilon(|\boldsymbol{g}'| - \varepsilon\kappa \cos\theta) \, d\widehat{\sigma}. \tag{2.10}$$

Clearly, Ω_ε and its complementary Ω'_ε are connected domains but they are not simply connected. To define a cut in Ω'_ε, we denote by Σ_0 the set $\boldsymbol{F}_0((0,1)^2 \times \{0\})$ and $\partial\Sigma_0 = \boldsymbol{F}_0((0,1) \times \{1\} \times \{0\})$. Let Σ' denote a smooth simple surface that has $\partial\Sigma_0$ as a boundary and such that the surface $\Sigma = \Sigma' \cup \Sigma_0$ is oriented and of class $\mathcal{C}^1$ (cf. [5]). We denote by Σ^+ (*resp.* Σ^-) the oriented surface with positive (*resp.* negative) orientation, and by n the unit normal on Σ directed from Σ^+ to Σ^-. If $w \in W^1(\mathbb{R}^3 \setminus \Sigma)$, we denote by $[w]_\Sigma$ the jump of w across Σ through n, *i.e.*

$$[w]_\Sigma = w_{|\Sigma^+} - w_{|\Sigma^-}.$$

3 Formulation of the problem and statement of the result

We consider the boundary value problem

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \dfrac{\partial u^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, & \\ \left[\dfrac{\partial u^\varepsilon}{\partial n}\right]_\Sigma = 0, & \end{cases} \tag{3.1}$$

where n_ε denotes the unit normal on Γ_ε pointing outward Ω'_ε and $\boldsymbol{n}$ is the unit normal on Σ oriented from Σ^+ toward Σ^-. The inductance coefficient is defined by

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\boldsymbol{x}. \tag{3.2}$$

We want to describe the asymptotic behaviour of u^ε and L^ε as $\varepsilon \to 0$.
We first exhibit a function that has the same singularity as might have the solution of Problem (3.1) (as $\varepsilon \to 0$). Let us define

$$\widehat{v}(s, \xi, \theta) = \frac{\theta}{2\pi} \widehat{\varphi}(\xi), \qquad (s, \xi, \theta) \in \widehat{\Omega},$$

where $\widehat{\varphi} \in \mathcal{C}^2(\mathbb{R})$ and such that

$$\widehat{\varphi}(\xi) = 1 \text{ for } 0 \leq \xi \leq \frac{1}{2}, \quad \widehat{\varphi}(\xi) = 0 \text{ for } \xi \geq \frac{3}{4}.$$

We then define $v : \mathbb{R}^3 \to \mathbb{R}$ by :

$$v(\boldsymbol{x}) = \begin{cases} \widehat{v}(\boldsymbol{F}_0^{-1}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \in \Omega_\delta, \\ 0 & \text{if } \boldsymbol{x} \in \Omega'_\delta. \end{cases}$$

Let us also define

$$\widehat{f}(s,\xi,\theta) = \frac{1}{2\pi a_0}\left(\frac{\kappa\,\sin\theta}{\delta\xi} - \frac{\tau^2\,\delta\,\xi\,\kappa\,\sin\theta}{a_0^2} - \frac{\partial}{\partial s}\left(\frac{\tau}{a_0}\right)\right)\widehat{\varphi},$$
$$+\frac{\theta}{2\pi\,a_0\,\delta^2\,\xi}(2a_0 - |\boldsymbol{g}'|)\,\widehat{\varphi}' + \frac{\theta}{2\pi\,\delta^2}\,\widehat{\varphi}'', \qquad (s,\xi,\theta)\in\widehat{\Omega},$$

$$f(\boldsymbol{x}) = \begin{cases} \widehat{f}(\boldsymbol{F}_0^{-1}(\boldsymbol{x})) & \text{if } \boldsymbol{x}\in\Omega_\delta,\\ 0 & \text{if } \boldsymbol{x}\in\Omega_\delta', \end{cases}$$

$$\varphi(\boldsymbol{x}) = \begin{cases} \widehat{\varphi}(\xi) & \text{if } \boldsymbol{x}\in\Omega_\delta, \text{ with } (s,\xi,\theta) = \boldsymbol{F}_0^{-1}(\boldsymbol{x}),\\ 0 & \text{if } \boldsymbol{x}\in\Omega_\delta'. \end{cases}$$

We have the following result.

Proposition 3.1. The function v is solution of

$$\begin{cases} \Delta v = f & \text{in } \mathbb{R}^3\setminus\Sigma,\\ [v]_\Sigma = \varphi, & \\ \left[\dfrac{\partial v}{\partial n}\right]_\Sigma = 0. & \end{cases} \tag{3.3}$$

Moreover, it satisfies

$$\frac{\partial v}{\partial n_\varepsilon} = 0 \qquad \text{on } \Gamma_\varepsilon. \tag{3.4}$$

Proof. The first equation in (3.3) follows readily from definitions of f and v. It remains to check the boundary conditions. On Σ_0', we have obviously

$$[v]_{\Sigma_0'} = \left[\frac{\partial v}{\partial n}\right]_{\Sigma_0'} = 0.$$

On Σ_0, we have

$$v_{|\Sigma_0^+} = \varphi,\ \ v_{|\Sigma_0^-} = 0,$$

whence $[v]_\Sigma = \varphi$. We also have, according to (2.6), (2.7),

$$\widehat{\nabla v}_{|\widehat{\Sigma}_0} = -\frac{\tau}{2\pi a_0}\widehat{\varphi}\,\boldsymbol{t} + \frac{1}{\delta}\,\widehat{\varphi}'\,\boldsymbol{\nu} + \frac{1}{2\pi\delta\xi}\,\widehat{\varphi}\,\boldsymbol{b} \qquad \text{for } \theta = 2\pi,$$
$$\widehat{\nabla v}_{|\widehat{\Sigma}_0} = -\frac{\tau}{2\pi a_0}\widehat{\varphi}\,\boldsymbol{t} + \frac{1}{2\pi\delta\xi}\,\widehat{\varphi}\,\boldsymbol{b} \qquad \text{for } \theta = 0,$$

with $\widehat{\Sigma}_0 = (0,1)^2$. The normal to Σ_0 is defined by

$$\widehat{\boldsymbol{n}} = \frac{1}{((|\boldsymbol{g}'| - \delta\xi\kappa)^2 + \delta^2\xi^2\tau^2)^{\frac{1}{2}}}\,((|\boldsymbol{g}'| - \delta\xi\kappa)\boldsymbol{b} - \delta\xi\tau\boldsymbol{t}).$$

Therefore

$$\widehat{\frac{\partial v}{\partial n}}_{|\Sigma_0} = \frac{\widehat{\varphi}}{2\pi((|\boldsymbol{g}'| - \delta\xi\kappa)^2 + \delta^2\xi^2\tau^2)^{\frac{1}{2}}}\left(\frac{|\boldsymbol{g}'| - \delta\xi\kappa}{\delta\xi} + \frac{\delta\xi\tau^2}{a_0}\right),$$

and then

$$\left[\frac{\partial v}{\partial n}\right]_{\Sigma_0} = 0.$$

We have, by (2.6)–(2.8),

$$\widehat{\nabla v} = \frac{1}{a_0}\Big(\frac{\partial \widehat{v}}{\partial s} - \tau \frac{\partial \widehat{v}}{\partial \theta}\Big)\,\boldsymbol{t} + \Big(\frac{\cos\theta}{\delta}\frac{\partial \widehat{v}}{\partial \xi} - \frac{\sin\theta}{\delta\xi}\frac{\partial \widehat{v}}{\partial \theta}\Big)\,\boldsymbol{\nu} + \Big(\frac{\sin\theta}{\delta}\frac{\partial \widehat{v}}{\partial \xi} + \frac{\cos\theta}{\delta\xi}\frac{\partial \widehat{v}}{\partial \theta}\Big)\,\boldsymbol{b}.$$

The normal to Γ_ε is parametrically represented by $-(\cos\theta\boldsymbol{\nu} + \sin\theta\boldsymbol{b})$. Then, since $\widehat{\varphi}'(\frac{\varepsilon}{\delta}) = 0$,

$$\widehat{\frac{\partial v}{\partial n_\varepsilon}}_{|\Gamma_\varepsilon} = -\frac{1}{\delta}\frac{\partial \widehat{v}}{\partial \xi}(s, \frac{\varepsilon}{\delta}, \theta) = -\frac{\theta}{2\pi\delta}\widehat{\varphi}'(\frac{\varepsilon}{\delta}) = 0.$$

We conclude that v is solution of Problem (3.3). □

Lemma 3.1. For any $1 \le p < 2$ we have

$$f \in L^p(\mathbb{R}^3),\ v \in L^\infty(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3 \setminus \Sigma).$$

Proof. Clearly $v \in L^\infty(\mathbb{R}^3)$. Let us calculate the L^p–norm of f. Using the mapping $\boldsymbol{F}_0^{-1}$, we have

$$\begin{aligned}\|f\|^p_{L^p(\mathbb{R}^3\setminus\Sigma)} &= \|f\|^p_{L^p(\Omega_\delta)}\\ &= \frac{1}{(2\pi)^p}\int_{\widehat{\Omega}}\Big|\frac{1}{a_0}\left(\frac{\kappa\,\sin\theta}{\delta\xi} - \frac{\tau^2\,\delta\,\xi\,\kappa\,\sin\theta}{a_0^2} - \frac{\partial}{\partial s}\left(\frac{\tau}{a_0}\right)\right)\widehat{\varphi}\\ &\qquad + \frac{\theta}{a_0\,\xi\,\delta^2}(2a_0 - |\boldsymbol{g}'|)\,\widehat{\varphi}' + \frac{\theta}{\delta^2}\,\widehat{\varphi}''\Big|^p\,\delta^2\,a_0\,\xi\,d\widehat{\boldsymbol{x}}.\end{aligned}$$

Owing to (2.2) and to the fact that $\widehat{\varphi}$ is of class $\mathcal{C}^2$, we deduce that the above integral is finite provided that $1 \le p < 2$.
Using (2.6)–(2.8), we get

$$\|\nabla v\|^p_{\boldsymbol{L}^p(\mathbb{R}^3\setminus\Sigma)} = \frac{\delta^2}{(2\pi)^p}\int_{\widehat{\Omega}} a_0\,\xi\left|\frac{\theta^2}{\delta^2}(\widehat{\varphi}')^2 + \left(\frac{1}{\delta^2\xi^2} + \frac{\tau^2}{a_0^2}\right)\widehat{\varphi}^2\right|^{\frac{p}{2}}\,d\widehat{\boldsymbol{x}}.$$

With the same argument as for f, we deduce that the above integral is finite iff $1 \le p < 2$. □

Let us now set $w^\varepsilon = u^\varepsilon - v$. We have by subtracting (3.3) from (3.1),

$$\begin{cases} -\Delta w^\varepsilon = f & \text{in } \Omega'_\varepsilon \setminus \Sigma,\\ \dfrac{\partial w^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon,\\ [w^\varepsilon]_\Sigma = 1 - \varphi, & \\ \left[\dfrac{\partial w^\varepsilon}{\partial n}\right]_\Sigma = 0. & \end{cases} \tag{3.5}$$

We note here that Problem (3.5) differs from (3.1) by the value of the jump of the solution across Σ and by the presence of a right-hand side f. However, we notice that $(1-\varphi)$ vanishes in a neighborhood of $\partial\Sigma$ and then, for Problem (3.5), the jump of w^ε vanishes in a neighborhood of $\partial\Sigma$.
Now, to study the asymptotic behaviour of w^ε and L^ε as $\varepsilon \to 0$ we consider the following decomposition. Let w_1 denote the solution of

$$\begin{cases} \Delta w_1 = 0 & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [w_1]_\Sigma = 1-\varphi, & \\ \left[\dfrac{\partial w_1}{\partial n}\right]_\Sigma = 0, & \\ w_1(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty. \end{cases} \tag{3.6}$$

Using [4], p. 654, and the fact that $(1-\varphi)$ vanishes in a neighborhood of $\partial\Sigma$, we see that Problem (3.6) has a unique solution in $W^1(\mathbb{R}^3 \setminus \Sigma)$ given by

$$w_1(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Sigma (1-\varphi(\boldsymbol{y}))\frac{\boldsymbol{n}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,d\sigma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Sigma. \tag{3.7}$$

Then we write $w^\varepsilon = w_1 + w_2^\varepsilon$, where the function w_2^ε is solution of the exterior Neumann problem :

$$\begin{cases} -\Delta w_2^\varepsilon = f & \text{in } \Omega'_\varepsilon, \\ \dfrac{\partial w_2^\varepsilon}{\partial n_\varepsilon} = -\dfrac{\partial w_1}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon, \\ w_2^\varepsilon(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty. \end{cases} \tag{3.8}$$

We have the following result.

Lemma 3.2. Problem (3.8) admits a unique solution $w_2^\varepsilon \in W^1(\Omega'_\varepsilon)$.

Proof. Differentiating (3.7), we obtain for $\boldsymbol{x} \in \Gamma_\varepsilon$:

$$\begin{aligned} \frac{\partial w_1}{\partial n_\varepsilon}(\boldsymbol{x}) = & \frac{1}{4\pi}\int_\Sigma (1-\varphi(\boldsymbol{y}))\frac{\boldsymbol{n}_\varepsilon(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,d\sigma(\boldsymbol{y}) \\ & - \frac{3}{4\pi}\int_\Sigma (1-\varphi(\boldsymbol{y}))\frac{(\boldsymbol{n}_\varepsilon(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y}))\,(\boldsymbol{n}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y}))}{|\boldsymbol{x}-\boldsymbol{y}|^5}\,d\sigma(\boldsymbol{y}). \end{aligned}$$

Owing to the definition of φ, the integrals over Σ reduce to those over $\widetilde{\Sigma}$ where

$$\widetilde{\Sigma} = \boldsymbol{\Phi}_\varepsilon((0,1)\times(\frac{1}{2},1)) \cup \Sigma'.$$

So, for $\boldsymbol{x} \in \Gamma_\varepsilon$ and $\boldsymbol{y} \in \widetilde{\Sigma}$, $|\boldsymbol{x}-\boldsymbol{y}| \geq \frac{\delta}{4}$ since ε is chosen not greater than $\frac{\delta}{2}$. Therefore

$$\left\|\frac{\partial w_1}{\partial n_\varepsilon}\right\|_{L^\infty(\Gamma_\varepsilon)} \leq C, \tag{3.9}$$

and, since $f_{|\Omega'_\varepsilon} \in L^2(\Omega'_\varepsilon)$, then Problem (3.8) is a classical exterior Neumann problem which admits a unique solution $w_2^\varepsilon \in W^1(\Omega'_\varepsilon)$, see [3], p. 343. $\square$

Let finally w_2 denote the unique solution in $W^1(\mathbb{R}^3)$ of

$$\begin{cases} -\Delta w_2 = f & \text{in } \mathbb{R}^3, \\ w_2(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), & |\boldsymbol{x}| \to \infty. \end{cases} \tag{3.10}$$

As it is classical (see [7] for instance) the function w_2 is given by

$$w_2(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\boldsymbol{y}, \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

Summarizing the decomposition process of the solution to Problem (3.1), we have

$$u^\varepsilon = v + w_1 + w_2^\varepsilon \qquad \text{in } \Omega'_\varepsilon \setminus \Sigma,$$

where v, w_1 and w_2^ε are solutions of (3.3), (3.6) and (3.8) respectively.
We now state our main result.

Theorem 3.1. Let u^ε be the solution of Problem (3.1) and let L^ε be the inductance coefficient defined by (3.2). Let u be the function defined in $\mathbb{R}^3 \setminus \Sigma$ by $u = v + w_1 + w_2$, where v, w_1 and w_2 are solutions of (3.3), (3.6) and (3.10) respectively. Then for any $\eta > 0$:

$$\|u - u^\varepsilon\|_{W^1(\Omega'_\varepsilon)} = O(\varepsilon^{\frac{5}{6}-\eta}), \tag{3.11}$$

$$\begin{aligned} L^\varepsilon = & -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + L' - \int_{\mathbb{R}^3} f(w_1+w_2)\, d\boldsymbol{x} \\ & + \int_\Sigma (1-\varphi)\left(\frac{\partial w_1}{\partial n} + \frac{\partial w_2}{\partial n} + 2\frac{\partial v}{\partial n}\right) d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}), \end{aligned} \tag{3.12}$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{1}{4\pi^2}\int_{\widehat{\Omega}} \left(a_0 \xi \theta^2 (\widehat{\varphi}')^2 + \frac{\delta^2 \xi \tau^2}{a_0}\widehat{\varphi}^2\right) d\widehat{\boldsymbol{x}} + \ell_\gamma \int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{2\pi\xi}\, d\xi.$$

The next section is devoted to the proof of this result.

4 Proof of Theorem 3.1

Let us first give estimates of the trace on Γ_ε for functions of $W^1(\Omega'_\varepsilon)$ or $W^{1,p}(\Omega'_\varepsilon)$, $\frac{3}{2} < p < 2$.

Lemma 4.1. There is a constant C, independent of ε, such that :

$$\|\psi\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{\frac{1}{2}}\,|\ln\varepsilon|^{\frac{1}{2}}\,\|\psi\|_{W^1(\Omega'_\varepsilon)} \qquad \text{for all } \psi \in W^1(\Omega'_\varepsilon), \tag{4.1}$$

$$\begin{aligned} & \|\psi\|_{L^2(\Gamma_\varepsilon)} \le C\left(\varepsilon^{\frac{1}{2}}\,\|\psi\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{4}{3}-\frac{2}{p}}\,\|\nabla\psi\|_{\boldsymbol{L}^p(\Lambda^\delta_\varepsilon)}\right) \\ & \qquad \text{for all } \psi \in W^{1,p}(\Omega'_\varepsilon) \text{ with compact support, } \frac{3}{2} < p < 2. \end{aligned} \tag{4.2}$$

Proof. Let $\psi \in \mathcal{C}^1(\overline{\Omega}'_\varepsilon)$ with compact support and let $\widehat{\psi} : \widehat{\Omega} \to \mathbb{R}$ defined by

$$\widehat{\psi}(\widehat{\boldsymbol{x}}) = \psi(\boldsymbol{F}_\varepsilon(\widehat{\boldsymbol{x}})), \qquad \widehat{\boldsymbol{x}} \in \widehat{\Omega}.$$

Let us first prove (4.1). We have

$$\widehat{\psi}(s,0,\theta) = \widehat{\psi}(s,1,\theta) - \int_0^1 \frac{\partial \widehat{\psi}}{\partial \xi}(s,\xi,\theta)\, d\xi, \qquad (s,\theta) \in \widehat{\Gamma}.$$

Consequently,

$$|\widehat{\psi}(s,0,\theta)|^2 \leq 2|\widehat{\psi}(s,1,\theta)|^2 + 2\left(\int_0^1 \frac{\partial \widehat{\psi}}{\partial \xi}(s,\xi,\theta)\, d\xi\right)^2, \tag{4.3}$$

and, using the Cauchy–Schwarz inequality and (2.2) :

$$\begin{aligned}
|\widehat{\psi}(s,0,\theta)|^2 &\leq 2|\widehat{\psi}(s,1,\theta)|^2 + 2\left(\int_0^1 \frac{1}{a_\varepsilon r_\varepsilon}\, d\xi\right)\left(\int_0^1 a_\varepsilon r_\varepsilon \Big|\frac{\partial \widehat{\psi}}{\partial \xi}\Big|^2 d\xi\right) \\
&\leq 2|\widehat{\psi}(s,1,\theta)|^2 + 2C_1\left(\int_0^1 \frac{1}{r_\varepsilon}\, d\xi\right)\left(\int_0^1 a_\varepsilon r_\varepsilon \Big|\frac{\partial \widehat{\psi}}{\partial \xi}\Big|^2 d\xi\right) \\
&\leq 2|\widehat{\psi}(s,1,\theta)|^2 + C_2\, |\ln \varepsilon| \int_0^1 a_\varepsilon r_\varepsilon \Big|\frac{\partial \widehat{\psi}}{\partial \xi}\Big|^2 d\xi,
\end{aligned} \tag{4.4}$$

for $(s,\theta) \in \widehat{\Gamma}$. Since by (2.10),

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} = \varepsilon \int_{\widehat{\Gamma}} \alpha_\varepsilon(s,\theta) |\widehat{\psi}(s,0,\theta)|^2\, ds\, d\theta, \tag{4.5}$$

$$\|\psi\|^2_{L^2(\Gamma_\delta)} = \delta \int_{\widehat{\Gamma}} \alpha_\delta(s,\theta) |\widehat{\psi}(s,1,\theta)|^2\, ds\, d\theta, \tag{4.6}$$

with

$$\alpha_\varepsilon(s,\theta) = a_\varepsilon(s,0,\theta), \quad \alpha_\delta(s,\theta) = a_\delta(s,1,\theta).$$

We deduce from (4.4), after multiplication by $\varepsilon\alpha_\varepsilon$ and integration in s,θ,

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} \leq 2\varepsilon \int_{\widehat{\Gamma}} \alpha_\varepsilon |\widehat{\psi}(s,1,\theta)|^2\, ds\, d\theta + C_2 \varepsilon |\ln \varepsilon| \int_{\widehat{\Omega}} \alpha_\varepsilon a_\varepsilon r_\varepsilon \Big|\frac{\partial \widehat{\psi}}{\partial \xi}\Big|^2 d\widehat{\boldsymbol{x}}.$$

Using (2.2) and the estimates $0 < C_3' \leq \alpha_\varepsilon,\, \alpha_\delta \leq C_4'$, we get

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} \leq C_3 \varepsilon\, \delta \int_{\widehat{\Gamma}} \alpha_\delta |\widehat{\psi}(s,1,\theta)|^2\, ds\, d\theta + C_4 \varepsilon\, |\ln \varepsilon| \int_{\widehat{\Omega}} a_\varepsilon r_\varepsilon \Big|\frac{\partial \widehat{\psi}}{\partial \xi}\Big|^2 d\widehat{\boldsymbol{x}}.$$

But (2.9) yields

$$\|\nabla \psi\|^2_{\boldsymbol{L}^2(\Lambda^\delta_\varepsilon)} = (\delta - \varepsilon) \int_{\widehat{\Omega}} \left(\frac{r_\varepsilon}{a_\varepsilon}\Big(\frac{\partial \widehat{\psi}}{\partial s} - \tau \frac{\partial \widehat{\psi}}{\partial \theta}\Big)^2 + \frac{r_\varepsilon a_\varepsilon}{(\delta - \varepsilon)^2}\Big(\frac{\partial \widehat{\psi}}{\partial \xi}\Big)^2 + \frac{a_\varepsilon}{r_\varepsilon}\Big(\frac{\partial \widehat{\psi}}{\partial \theta}\Big)^2 \right) d\widehat{\boldsymbol{x}}.$$

Therefore

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} \le C_3\varepsilon\|\psi\|^2_{L^2(\Gamma_\delta)} + C_5\varepsilon|\ln\varepsilon|\,\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Lambda^\delta_\varepsilon)}.$$

Using the trace inequality and the fact that the support of ψ is compact, we obtain

$$\begin{aligned}\|\psi\|^2_{L^2(\Gamma_\varepsilon)} &\le (C_3\,C_6\,\varepsilon + C_5\varepsilon|\ln\varepsilon|)\,\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega'_\varepsilon)}\\ &\le C_7\,\varepsilon\,|\ln\varepsilon|\,\|\nabla\psi\|^2_{W^1(\Omega'_\varepsilon)}.\end{aligned}$$

By density, (4.1) follows.
Let us now prove (4.2). We have

$$\begin{aligned}|\widehat{\psi}(s,0,\theta)|^2 &= |\widehat{\psi}(s,1,\theta)|^2 - \int_0^1 \frac{\partial}{\partial\xi}(\widehat{\psi})^2\,d\xi\\ &= |\widehat{\psi}(s,1,\theta)|^2 - 2\int_0^1 \widehat{\psi}\frac{\partial\widehat{\psi}}{\partial\xi}\,d\xi.\end{aligned}$$

Multiplying by ε and integrating in s,θ, we get

$$\varepsilon\int_0^1\int_0^{2\pi}|\widehat{\psi}(s,0,\theta)|^2\,d\theta\,ds = \varepsilon\int_0^1\int_0^{2\pi}|\widehat{\psi}(s,1,\theta)|^2\,d\theta\,ds - 2\varepsilon\int_{\widehat{\Omega}}\widehat{\psi}\frac{\partial\widehat{\psi}}{\partial\xi}\,d\widehat{\boldsymbol{x}}.$$

Using (4.5), (4.6) and (2.2), we get

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} \le C_8\varepsilon\,\|\psi\|^2_{L^2(\Gamma_\delta)} + C_9\varepsilon\left|\int_{\widehat{\Omega}}\widehat{\psi}\frac{\partial\widehat{\psi}}{\partial\xi}\,d\widehat{\boldsymbol{x}}\right|. \tag{4.7}$$

To estimate the integral in the previous relationship we use the Hölder inequality

$$\left|\int_{\widehat{\Omega}}\widehat{\psi}\frac{\partial\widehat{\psi}}{\partial\xi}\,d\widehat{\boldsymbol{x}}\right| \le \Big(\int_{\widehat{\Omega}} r_\varepsilon|\widehat{\psi}|^q\,d\widehat{\boldsymbol{x}}\Big)^{\frac{1}{q}}\Big(\int_{\widehat{\Omega}} r_\varepsilon\Big|\frac{\partial\widehat{\psi}}{\partial\xi}\Big|^p\,d\widehat{\boldsymbol{x}}\Big)^{\frac{1}{p}}\Big(\int_{\widehat{\Omega}} r_\varepsilon^{1-m}\,d\widehat{\boldsymbol{x}}\Big)^{\frac{1}{m}},$$

where $q=\frac{3p}{3-p}$ and m is such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{m}=1$, *i.e.*, $m=\frac{3p}{4p-6}$. Using (2.6)–(2.8), we have

$$\begin{aligned}\|\nabla\psi\|_{\boldsymbol{L}^p(\Lambda^\delta_\varepsilon)} = \Bigg(\int_{\widehat{\Omega}}(\delta-\varepsilon)a_\varepsilon r_\varepsilon\Bigg(&\frac{1}{a_\varepsilon^2}\Big(\frac{\partial\widehat{\psi}}{\partial s}-\tau\frac{\partial\widehat{\psi}}{\partial\theta}\Big)^2 + \frac{1}{r_\varepsilon^2}\Big(\frac{\partial\widehat{\psi}}{\partial\theta}\Big)^2\\ &+\frac{1}{(\delta-\varepsilon)^2}\Big(\frac{\partial\widehat{\psi}}{\partial\xi}\Big)^2\Bigg)^{\frac{p}{2}}\,d\widehat{\boldsymbol{x}}\Bigg)^{\frac{1}{p}}.\end{aligned}$$

Using (2.2), we then have

$$\begin{aligned}\left|\int_{\widehat{\Omega}}\widehat{\psi}\frac{\partial\widehat{\psi}}{\partial\xi}\,d\widehat{\boldsymbol{x}}\right| &\le C_{10}\,\|\psi\|_{L^q(\Lambda^\delta_\varepsilon)}\,\|\nabla\psi\|_{\boldsymbol{L}^p(\Lambda^\delta_\varepsilon)}\Big(\int_{\widehat{\Omega}} r_\varepsilon^{1-m}\,d\widehat{\boldsymbol{x}}\Big)^{\frac{1}{m}}\\ &\le C_{11}\varepsilon^{\frac{2-m}{m}}\,\|\psi\|_{L^q(\Lambda^\delta_\varepsilon)}\,\|\nabla\psi\|_{\boldsymbol{L}^p(\Lambda^\delta_\varepsilon)}.\end{aligned}$$

We note here that $m > 2$. Then the imbedding of $W^{1,p}(\Lambda_\varepsilon^\delta)$ into $L^q(\Lambda_\varepsilon^\delta)$ implies

$$\left| \int_{\widehat{\Omega}} \widehat{\psi} \frac{\partial \widehat{\psi}}{\partial \xi} \, d\widehat{\boldsymbol{x}} \right| \le C_{12} \, \varepsilon^{\frac{2-m}{m}} \, \|\nabla \psi\|^2_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)} = C_{12} \, \varepsilon^{\frac{5}{3} - \frac{4}{p}} \, \|\nabla \psi\|^2_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)}.$$

Putting this estimate into (4.7) yields

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} \le C_8 \, \varepsilon \, \|\psi\|^2_{L^2(\Gamma_\delta)} + C_9 \, C_{12} \, \varepsilon^{\frac{8}{3} - \frac{4}{p}} \, \|\nabla \psi\|^2_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)}.$$

Using the trace inequality

$$\|\psi\|_{L^2(\Gamma_\delta)} \le C_{13} \, \|\psi\|_{W^{1,p}(\Omega'_\delta)},$$

we get

$$\begin{aligned} \|\psi\|^2_{L^2(\Gamma_\varepsilon)} &\le C \left(\varepsilon \, \|\psi\|^2_{W^{1,p}(\Omega'_\delta)} + \varepsilon^{\frac{8}{3} - \frac{4}{p}} \, \|\nabla \psi\|^2_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)} \right) \\ &\le C \left(\varepsilon \, \|\psi\|^2_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{8}{3} - \frac{4}{p}} \, \|\nabla \psi\|^2_{\boldsymbol{L}^p(\Lambda_\varepsilon^\delta)} \right) \end{aligned}$$

The conclusion of the lemma follows by density. □

4.1 Proof of Estimate (3.11)

Let $\widetilde{w}_2^\varepsilon = w_2^\varepsilon - w_2$. Clearly $\widetilde{w}_2^\varepsilon = u^\varepsilon - u$, $\widetilde{w}_2^\varepsilon \in W^1(\Omega'_\varepsilon)$ and it satisfies

$$\begin{cases} \Delta \widetilde{w}_2^\varepsilon = 0 & \text{in } \Omega'_\varepsilon, \\ \dfrac{\partial \widetilde{w}_2^\varepsilon}{\partial n_\varepsilon} = -\dfrac{\partial w_1}{\partial n_\varepsilon} - \dfrac{\partial w_2}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon, \\ \widetilde{w}_2^\varepsilon(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), & |\boldsymbol{x}| \to +\infty. \end{cases} \tag{4.8}$$

Using the variational formulation associated with (4.8), Cauchy–Schwarz inequality and Estimate (4.1), we deduce

$$\begin{aligned} \int_{\Omega'_\varepsilon} |\nabla \widetilde{w}_2^\varepsilon|^2 \, d\boldsymbol{x} &= \int_{\Gamma_\varepsilon} \left(\frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right) \widetilde{w}_2^\varepsilon \, d\sigma \\ &\le \left\| \frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \|\widetilde{w}_2^\varepsilon\|_{L^2(\Gamma_\varepsilon)} \\ &\le C \, \varepsilon^{\frac{1}{2}} \, |\ln \varepsilon|^{\frac{1}{2}} \left(\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} + \left\| \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \right) \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)}. \end{aligned} \tag{4.9}$$

Using (3.9), we have

$$\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \le C \, (\operatorname{meas} \Gamma_\varepsilon)^{\frac{1}{2}} \le C_1 \varepsilon^{\frac{1}{2}}. \tag{4.10}$$

To estimate $\dfrac{\partial w_2}{\partial n_\varepsilon}$, we use standard regularity results for elliptic problems, see [3], p. 343, to deduce, since $f \in L^p(\mathbb{R}^3)$ for $p < 2$, that $w_2 \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)$. Then we apply Estimate (4.2) to the function $u = \dfrac{\partial w_2}{\partial x_i}$, $1 \le i \le 3$ with $p = 2 - \eta$, $0 < \eta < \frac{1}{2}$,

$$\left\| \frac{\partial w_2}{\partial x_i} \right\|_{L^2(\Gamma_\varepsilon)} \le C \left(\varepsilon^{\frac{1}{2}} \left\| \frac{\partial w_2}{\partial x_i} \right\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{1}{3} - \frac{\eta}{2-\eta}} \left\| \frac{\partial}{\partial x_i} \nabla w_2 \right\|_{\boldsymbol{L}^p(\Lambda^\delta_\varepsilon)} \right).$$

Since both norms on the right–hand side of the above inequality are uniformly bounded and since the outward unit normal n_ε is uniformly bounded we obtain

$$\left\| \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \le C \varepsilon^{\frac{1}{3} - \frac{\eta}{2-\eta}}. \tag{4.11}$$

Reporting (4.10) and (4.11) into (4.9) and using the inequality $|\ln \varepsilon| \le C\varepsilon^{-2\eta}$, we get

$$\int_{\Omega'_\varepsilon} |\nabla \widetilde{w}^\varepsilon_2|^2 \, d\boldsymbol{x} \le C_1 \, \varepsilon^{\frac{5}{6} - \frac{\eta}{2-\eta} - \eta} \, \|\nabla \widetilde{w}^\varepsilon_2\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)}.$$

Therefore

$$\|\nabla \widetilde{w}^\varepsilon_2\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \le C_2 \varepsilon^{\frac{5}{6} - \eta} \qquad \text{for all } \eta > 0.$$

$\square$

4.2 Proof of Estimate (3.12)

To prove (3.12) we need the following lemmas.

Lemma 4.2. We have for all $\eta > 0$,

$$L^\varepsilon = \int_{\Omega'_\varepsilon} |\nabla v|^2 \, d\boldsymbol{x} - \int_{\mathbb{R}^3} f w \, d\boldsymbol{x} + \int_\Sigma (1 - \varphi) \left(\frac{\partial w}{\partial n} + 2 \frac{\partial v}{\partial n} \right) d\sigma + O(\varepsilon^{\frac{5}{6} - \eta}), \tag{4.12}$$

where $w = w_1 + w_2$.

Proof. Using the decomposition $u^\varepsilon = v + w^\varepsilon = v + w_1 + w^\varepsilon_2$ it follows :

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla v|^2 \, d\boldsymbol{x} + \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla w^\varepsilon|^2 \, d\boldsymbol{x} + 2 \int_{\Omega'_\varepsilon \setminus \Sigma} \nabla v \cdot \nabla w^\varepsilon \, d\boldsymbol{x}.$$

The estimation of the last two integrals can be achieved as follows. We use (3.5) and the Green's formula to obtain

$$\begin{aligned} \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla w^\varepsilon|^2 \, d\boldsymbol{x} &= - \int_{\Omega'_\varepsilon \setminus \Sigma} w^\varepsilon \Delta w^\varepsilon \, d\boldsymbol{x} - \int_{\Gamma_\varepsilon} w^\varepsilon \frac{\partial w^\varepsilon}{\partial n_\varepsilon} \, d\sigma + \int_\Sigma (1 - \varphi) \frac{\partial w^\varepsilon}{\partial n_\varepsilon} \, d\sigma \\ &= \int_{\Omega'_\varepsilon} f w^\varepsilon \, d\boldsymbol{x} + \int_\Sigma (1 - \varphi) \frac{\partial w^\varepsilon}{\partial n} \, d\sigma. \end{aligned}$$

Similarly, we use (3.3) to get

$$\begin{aligned}\int_{\Omega'_\varepsilon\setminus\Sigma} \nabla v\cdot\nabla w^\varepsilon\,d\boldsymbol{x} &= -\int_{\Omega'_\varepsilon\setminus\Sigma} w^\varepsilon \Delta v\,d\boldsymbol{x} - \int_{\Gamma_\varepsilon} w^\varepsilon \frac{\partial v}{\partial n_\varepsilon}\,d\sigma + \int_\Sigma (1-\varphi)\frac{\partial v}{\partial n}\,d\sigma \\ &= -\int_{\Omega'_\varepsilon} f w^\varepsilon\,d\boldsymbol{x} + \int_\Sigma (1-\varphi)\frac{\partial v}{\partial n}\,d\sigma.\end{aligned}$$

Then

$$L^\varepsilon = \int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w^\varepsilon\,d\boldsymbol{x} + \int_\Sigma (1-\varphi)\left(\frac{\partial w^\varepsilon}{\partial n} + 2\frac{\partial v}{\partial n}\right)d\sigma. \tag{4.13}$$

We can now estimate the error between the above expression of L^ε and the desired one. We have, with $w = w_1 + w_2$,

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f w\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w^\varepsilon\,d\boldsymbol{x}\right| &= \left|\int_{\mathbb{R}^3} f w_1\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w_1\,d\boldsymbol{x} + \int_{\mathbb{R}^3} f w_2\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w_2^\varepsilon\,d\boldsymbol{x}\right| \\ &\le \left|\int_{\Omega_\varepsilon} f w_1\,d\boldsymbol{x}\right| + \left|\int_{\mathbb{R}^3} f w_2\,d\boldsymbol{x} - \int_{\Omega'_\varepsilon} f w_2\,d\boldsymbol{x}\right| \\ &\qquad + \left|\int_{\Omega'_\varepsilon} f (w_2 - w_2^\varepsilon)\,d\boldsymbol{x}\right| \\ &\le \left|\int_{\Omega_\varepsilon} f w_1\,d\boldsymbol{x}\right| + \left|\int_{\Omega_\varepsilon} f w_2\,d\boldsymbol{x}\right| + \left|\int_{\Omega'_\varepsilon} f (w_2 - w_2^\varepsilon)\,d\boldsymbol{x}\right|.\end{aligned}$$

For $1 \le p < 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, we have thanks to Lemma 3.1 and since $w_2 \in W^{2,p}(\Omega_\delta) \subset L^\infty(\Omega_\delta)$,

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} f w_2\,d\boldsymbol{x}\right| &\le \|f\|_{L^p(\Omega_\varepsilon)}\,\|w_2\|_{L^q(\Omega_\varepsilon)} \\ &\le \|f\|_{L^p(\Omega_\varepsilon)}\,\|w_2\|_{L^\infty(\Omega_\varepsilon)}\,(\operatorname{meas}\Omega_\varepsilon)^{\frac{1}{q}} \\ &\le C\varepsilon^{\frac{2}{q}}.\end{aligned}$$

We also have, since $1-\varphi = 0$ in a neighborhood of $\partial\Sigma$ and then $w_1 \in H^2(\Omega_{\frac{\delta}{2}}) \subset L^\infty(\Omega_{\frac{\delta}{2}})$,

$$\left|\int_{\Omega_\varepsilon} f w_1\,d\boldsymbol{x}\right| \le \|f\|_{L^p(\Omega_\varepsilon)}\,\|w_1\|_{L^\infty(\Omega_\varepsilon)}\,(\operatorname{meas}\Omega_\varepsilon)^{\frac{1}{q}} \le C\,\varepsilon^{\frac{2}{q}}.$$

In addition, since $w_2 - w_2^\varepsilon = u - u^\varepsilon$,

$$\left|\int_{\Omega'_\varepsilon} f (w_2 - w_2^\varepsilon)\,d\boldsymbol{x}\right| \le \|f\|_{L^p(\Omega'_\varepsilon)}\,\|u - u^\varepsilon\|_{L^q(\Omega'_\varepsilon)}.$$

Choosing p so that $q < \frac{12}{5}$ and using (3.11), we obtain

$$\left|\int_{\Omega_\varepsilon} f w_1 \, d\boldsymbol{x}\right| + \left|\int_{\Omega_\varepsilon} f w_2 \, d\boldsymbol{x}\right| + \left|\int_{\Omega'_\varepsilon} f(w_2 - w_2^\varepsilon) \, d\boldsymbol{x}\right| \le C\,\varepsilon^{\frac{5}{6}-\eta},$$

for any $\eta > 0$. Now we have to estimate the difference of the two integrals over Σ in (4.13) and in (4.12). From (4.8), (3.10) and the identity $\widetilde{w}_2^\varepsilon = w_2^\varepsilon - w_2$, we deduce

$$\begin{aligned}\int_\Sigma (1-\varphi)\left(\frac{\partial w^\varepsilon}{\partial n} - \frac{\partial w}{\partial n}\right) d\sigma &= \int_\Sigma (1-\varphi)\frac{\partial \widetilde{w}_2^\varepsilon}{\partial n}\, d\sigma \\ &= \int_{\Omega'_\varepsilon\setminus\Sigma} \nabla w_2 \cdot \nabla \widetilde{w}_2^\varepsilon \, d\boldsymbol{x} + \int_{\Gamma_\varepsilon} w_2 \left(\frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon}\right) d\sigma.\end{aligned}$$

Then, using estimates (4.1), (4.10) and (4.11) we get for any $0 < \eta \le \frac{1}{2}$,

$$\begin{aligned}\left|\int_\Sigma (1-\varphi)\left(\frac{\partial w^\varepsilon}{\partial n} - \frac{\partial w}{\partial n}\right) d\sigma\right| &\le \|\nabla w_2\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \\ &\quad + \|w_2\|_{L^2(\Gamma_\varepsilon)} \left(\left\|\frac{\partial w_1}{\partial n_\varepsilon}\right\|_{L^2(\Gamma_\varepsilon)} + \left\|\frac{\partial w_2}{\partial n_\varepsilon}\right\|_{L^2(\Gamma_\varepsilon)}\right) \\ &\le \|\nabla w_2\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} \\ &\quad + C\varepsilon^{\frac{1}{2}}\,|\ln\varepsilon|^{\frac{1}{2}}\,\|w_2\|_{W^1(\Omega'_\varepsilon)}(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{3}-\frac{\eta}{2-\eta}}) \\ &\le C_1\left(\|\nabla \widetilde{w}_2^\varepsilon\|_{\boldsymbol{L}^2(\Omega'_\varepsilon)} + C\,|\ln\varepsilon|^{\frac{1}{2}}(\varepsilon + \varepsilon^{\frac{5}{6}-\frac{\eta}{2-\eta}})\right).\end{aligned} \tag{4.14}$$

Using the identity $\widetilde{w}_2^\varepsilon = u^\varepsilon - u$ and (3.11), we get

$$\left|\int_\Sigma (1-\varphi)\left(\frac{\partial w^\varepsilon}{\partial n} - \frac{\partial w}{\partial n}\right) d\sigma\right| \le C_2\,\varepsilon^{\frac{5}{6}-\eta}, \tag{4.15}$$

for any $\eta > 0$. Then we obtain the lemma from (4.13)–(4.15) □

Lemma 4.3. We have

$$\int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2 \, d\boldsymbol{x} = -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + L' + O(\varepsilon),$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{1}{4\pi^2}\int_{\widehat{\Omega}}\left(a_0\xi\theta^2(\widehat{\varphi}')^2 + \frac{\delta^2\xi\tau^2}{a_0}\widehat{\varphi}^2\right) d\widehat{\boldsymbol{x}} + \frac{\ell_\gamma}{2\pi}\int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{\xi}\, d\xi.$$

Proof. Using the definition of v and the change of variable $\boldsymbol{x} = \boldsymbol{F}_0(\widehat{\boldsymbol{x}})$, it follows

$$\int_{\Omega'_\varepsilon\setminus\Sigma} |\nabla v|^2 \, d\boldsymbol{x} = \int_{\Lambda_\varepsilon^\delta} |\nabla v|^2 \, d\boldsymbol{x} = A_\varepsilon^\delta + B_\varepsilon^\delta$$

with

$$A_\varepsilon^\delta = \int_{\widehat{\Omega}_\varepsilon^\delta} \frac{a_0\xi\theta^2}{4\pi^2}(\widehat{\varphi}')^2\,d\widehat{\boldsymbol{x}} + \int_{\widehat{\Omega}_\varepsilon^\delta} \frac{\delta^2\xi\tau^2}{4\pi^2 a_0}\widehat{\varphi}^2\,d\widehat{\boldsymbol{x}},$$
$$B_\varepsilon^\delta = \int_{\widehat{\Omega}_\varepsilon^\delta} \frac{a_0}{4\pi^2\xi}\widehat{\varphi}^2\,d\widehat{\boldsymbol{x}},$$

where $\widehat{\Omega}_\varepsilon^\delta = (0,1)\times(\frac{\varepsilon}{\delta},1)\times(0,2\pi)$. Clearly, we can write

$$A_\varepsilon^\delta = \int_{\widehat{\Omega}} \frac{a_0\xi\theta^2}{4\pi^2}(\widehat{\varphi}')^2\,d\widehat{\boldsymbol{x}} + \int_{\widehat{\Omega}} \frac{\delta^2\xi\tau^2}{4\pi^2 a_0}\widehat{\varphi}^2\,d\widehat{\boldsymbol{x}} + O(\varepsilon). \tag{4.16}$$

Since $\widehat{\varphi}(\xi) = 1$ for $0 \le \xi \le \frac{1}{2}$, we can write B_ε^δ as

$$\begin{aligned}
B_\varepsilon^\delta &= \int_0^1\int_{\frac{\varepsilon}{\delta}}^{\frac{1}{2}}\int_0^{2\pi} \frac{a_0}{4\pi^2\xi}\,d\theta\,d\xi\,ds + \int_0^1\int_{\frac{1}{2}}^1\int_0^{2\pi} \frac{a_0}{4\pi^2\xi}\widehat{\varphi}^2\,d\theta\,d\xi\,ds\\
&= \frac{1}{2\pi}\left(\int_0^1 |\boldsymbol{g}'(s)|\,ds\right)\int_{\frac{\varepsilon}{\delta}}^{\frac{1}{2}} \frac{d\xi}{\xi} + \int_0^1\int_{\frac{1}{2}}^1\int_0^{2\pi} \frac{a_0}{4\pi^2\xi}\widehat{\varphi}^2\,d\theta\,d\xi\,ds\\
&= -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \int_0^1\int_{\frac{1}{2}}^1\int_0^{2\pi} \frac{a_0}{4\pi^2\xi}\widehat{\varphi}^2\,d\theta\,d\xi\,ds\\
&= -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \int_0^1\int_{\frac{1}{2}}^1\int_0^{2\pi} \frac{|\boldsymbol{g}'|}{4\pi^2\xi}\widehat{\varphi}^2\,d\theta\,d\xi\,ds\\
&\qquad - \int_0^1\int_{\frac{1}{2}}^1\int_0^{2\pi} \frac{\delta\xi\kappa\cos\theta}{4\pi^2\xi}\widehat{\varphi}^2\,d\theta\,d\xi\,ds.\\
&= -\frac{\ell_\gamma}{2\pi}\ln\varepsilon + \frac{\ell_\gamma}{2\pi}\ln\frac{\delta}{2} + \frac{\ell_\gamma}{2\pi}\int_{\frac{1}{2}}^1 \frac{\widehat{\varphi}^2}{\xi}\,d\xi.
\end{aligned}$$

From this and (4.16) follows the lemma. □

Estimate (3.12) follows immediately by combining Lemmas 4.2 and 4.3.

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